A Note on Some Stochastic Orders and Aging Properties in the Accelerated Life Model

Xiaohu Li
Department of Mechanical Engineering
University of Alberta
Edmondon, Alberta, T6G 2G8
Canada
xiaohul@ualberta.ca

Ming J. Zuo
Department of Mechanical Engineering
University of Alberta
Edmondon, Alberta, T6G 2G8
Canada
ming.zuo@ualberta.ca

Abstract

The accelerated life model with time-dependent scale transformation is dealt with. The relative degradation for the two distribution functions is investigated through some other stochastic orders which are different from the usual stochastic order. The preservation of some new nonparametric aging properties under general accelerated life models are studied as well.

1 Introduction

The accelerated life model (ALM) in terms of Cox and Oakes [1] with time-dependent scale transformation function $W(t)$ claims that

$$F_1(t) = F_0(W(t)), \tag{1}$$

where $F_0(t)$ is a continuous distribution function for the lifetime, say X_0, of an item functioning in a baseline (reference) environment. Assume that $W(0) = 0$, $W(t)$ is monotonically increasing and

$$\lim_{t\to 0} W(t) = 0, \qquad \lim_{t\to\infty} W(t) = \infty.$$

Thus $F_1(t)$ gives the distribution function of another lifetime, say Y. Assume further that $W(t)$ is continuous and differentiable in $[0,\infty)$, then there should exist $w(t) > 0$, $t \in (0,\infty)$, such that

$$W(t) = \int_0^t w(u)du. \tag{2}$$

For those $W(t) \geq t$, equations (1), (2) models the impact of a more severe than baseline environment. In applications, $W(t)$ is usually assumed to be linear for simplicity:

$$F_1(t) = F_0(wt).$$

This note aims to investigating preservation properties of the increasing convex order and the increasing concave order under the accelerate model. The preservation properties of both NBUC and NBU(2) are also derived.

2 Stochastic orders

One convenient manner which can be used to characterize the impact of certain accelerate model is to conduct stochastic comparison between the random lifetime of the unit operating in baseline environment and that of the unit operating in a severe environment. It is pointed out in *Finkelstein* (2001) that X_0 is stochastically smaller than X_1 ($X_0 \geq_{st} X_1$) if and only if

$$W(t) \geq t, \qquad \text{for all } t \in [0,\infty), \tag{3}$$

where the inequality (3) is understood in terms of stochastic ordering:

$$\bar{F}_0(t) \geq \bar{F}_1(t), \qquad \text{for all } t \in [0, \infty),$$

and $\bar{F}_i(t) = 1 - F_i(t)$ is the reliability functions of X_i, $i = 0, 1$.

In fact, other stochastic orders involving dispersion can also be used to model the difference between the original lifetime and that be accelerated. One stronger notion is the dispersive order (*Shaked and Shanthikumar*, 1994): X_0 is more dispersed than X_1 if, for all $0 < u < v < 1$,

$$F_1^{-1}(v) - F_1^{-1}(v) \leq F_0^{-1}(v) - F_0^{-1}(v). \tag{4}$$

The first result investigates the dispersive order between X_0 and X_1.

Theorem 2.1 If $W(t) - t$ is increasing, then $X_0 \geq_{disp} X_1$.

Two of the definitions of the comparative variability in literature are the increasing convex order and the increasing concave order(*Shaked and Shanthikumar*, 1998; *Müller and Stoyan*, 2002).

X_0 is larger than X_1 in the increasing convex order ($X_0 \geq_{icx} X_1$) if, for all $x \geq 0$,

$$\int_x^\infty \bar{F}_0(t)dt \geq \int_x^\infty \bar{F}_1(t)dt.$$

In *Ross* (1996), it is called the variability order and denoted by $X_0 \geq_v X_1$ there.

X_0 is larger than X_1 in the increasing concave order ($X_0 \geq_{icv} X_1$) if, for all $x \geq 0$,

$$\int_0^x \bar{F}_0(t)dt \geq \int_0^x \bar{F}_1(t)dt.$$

Recently, the following two stronger versions of order were proposed because of good application in theory of reliability and some other related areas.

X_0 is larger than X_1 in excess wealth order ($X_0 \geq_{ew} X_1$) if, for all $p \in (0, 1)$,

$$\int_{F_0^{-1}(p)}^\infty \bar{F}_0(t)dt \geq \int_{F_1^{-1}(p)}^\infty \bar{F}_1(t)dt. \tag{5}$$

where $F_i^{-1}(p)$ is the right continuous inverse function of F_i, $i = 0, 1$. In *Fernadez, Kochar and Singh* (1998), this is called the right spread order and denoted by

$$X_0 \geq_{RS} X_1.$$

Another one was introduced recently by *Kochar, Li and Shaked* (2002). X_0 is larger than X_1 in the total time on test transform order ($X_0 \geq_{ttt} X_1$) if, for all $p \in (0, 1)$,

$$\int_0^{F_0^{-1}(p)} \bar{F}_0(t)dt \geq \int_0^{F_1^{-1}(p)} \bar{F}_1(t)dt. \tag{6}$$

According to *Shaked and Shanthikumar* (1998, 1994), *Kochar and Carriér* (1997) and *Kochar, Li and Shaked* (2002), we have the following chain of implications.

$$\begin{array}{ccccc} X_0 \geq_{disp} X_1 & \Longrightarrow & X_0 \geq_{ew} X_1 & \Longrightarrow & X_0 \geq_{icx} X_1 \\ \Downarrow & & & & \\ X_0 \geq_{st} X_1 & \Longrightarrow & X_0 \geq_{ttt} X_1 & \Longrightarrow & X_0 \geq_{icv} X_1. \end{array}$$

Our main results in this section study the sufficient conditions which lead to the above stochastic orders between X_0 nd X_1.

Theorem 2.2 If $W(t) - t$ is increasing and convex, then $X_0 \geq_{ttt} X_1$.

Theorem 2.3 If $W(t) - t$ is increasing and concave, then $X_0 \geq_{ew} X_1$.

In consideration that the excess wealth order and the TTT transform order imply the increasing convex order and the increasing concave order, respectively, we have got the results pointed out by *Finkelstein* (2001).

Corollary 2.4 (1) If $W(t) - t$ is increasing and convex, then $X_0 \geq_{icx} X_1$.
(2) If $W(t) - t$ is increasing and concave, then $X_0 \geq_{icv} X_1$.

3 Non-parametric aging properties

In theory of reliability, non-parametric aging properties are usually used to model the wear-out process of a random life. For example, IFR, IFRA, NBU and NBUE. In *Finkelstein* (2001), the inheriting property of the above four aging properties under the accelerating model is investigated, and the following results are concluded.

1. Assume $W(t)$ is convex (concave). If X_0 is IFR (DFR) then X_1 is also IFR (DFR).

2. Assume $W(t)$ is star-shaped (anti-star-shaped). If X_0 is IFRA (DFRA) then X_1 is also IFRA (DFRA).

3. Assume $W(t)$ is super-additive (sub-additive). If X_0 is NBU (NWU) then X_1 is also NBU (NWU).

To measure the degree of aging properties, the following partial orders are proposed (*Barlow and Proschan*, 1981).

Definition 3.1 (1) If $F_0^{-1}F_1(t)$ is convex, then X_1 is said to be more IFR than X_0 and denoted by $X_0 \leq_c X_1$.
(2) If $F_0^{-1}F_1(t)$ is star-shaped, then X_1 is said to be more IFRA than X_0 and denoted by $X_0 \leq_* X_1$.
(3) If $F_0^{-1}F_1(t)$ is super-additive, then X_1 is said to be more NBU than X_0 and denoted by $X_0 \leq_{su} X_1$.

Since $W(t) = F_0^{-1}F_1(t)$, the next main result follows immediately.

Theorem 3.2 If $W(t)$ is convex (star-shaped, super-additive), then $X_0 \leq_c X_1$ ($X_0 \leq_* X_1$, $X_0 \leq_{su} X_1$).

Since X_0 is IFR (IFRA, NBU) if and only if

$$Z \leq_c X_0 \quad (Z \leq_* X_0, \quad Z \leq_{su} X_0),$$

where Z is an exponential random life. By Theorem 3.2, we have

$$X_0 \leq_c X_1 \quad (X_0 \leq_* X_1, \quad X_0 \leq_{su} X_1).$$

Now, by transitivity, it holds that

$$Z \leq_c X_1 \quad (Z \leq_* X_1, \quad Z \leq_{su} X_1),$$

and hence, X_1 is IFR (IFRA, NBU) also. Thus, Theorem 1, 2, 3 in *Finkelstein* (2001) are special cases of Theorem 3.2.

Various stochastic orders are utilized to define the NBU properties in literature. For example, X_0 is said to be NBUC (NWUC) (*Cao and Wang*, 1991) if, for all $t \geq 0$ and $x \geq 0$,

$$\int_0^x \bar{F}_0(s)ds \geq (\leq) \int_0^x \bar{F}_{0t}(s)ds,$$

where $\bar{F}_{0t}(s) = \bar{F}_0(t+s)/\bar{F}_0(t)$ is the residual lifetime of X_0 at age $t \geq 0$. X_0 is said to be NBU(2) (NWU(2)) (*Deshpande, Kochar and Singh*, 1986) if, for all $t \geq 0$ and $x \geq 0$,

$$\int_x^\infty \bar{F}_0(s)ds \geq (\leq) \int_x^\infty \bar{F}_{0t}(s)ds.$$

It is obvious that

$$IFR \Longrightarrow IFRA \Longrightarrow NBU \Longrightarrow NBUC(NBU(2)) \Longrightarrow NBUE.$$

As the other two results in this section, the next theorem obtain sufficient conditions for the accelerate model to inherit the NBUC and NBU(2) properties of the original random life.

Theorem 3.3 (1) If X_0 is NBUC and $W(t)$ is increasing and concave, X_1 is also NBUC.
(2) If X_1 is NBUC and $W(t)$ is increasing and convex, then X_0 is also NBUC.

Theorem 3.4 (1) If X_0 is NWU(2) and $W(t)$ is increasing and convex, X_1 is also NWU(2).
(2) If X_1 is NWU(2) and $W(t)$ is increasing and concave, then X_0 is also NWU(2).

References

Barlow, R. E. and Proschan, F. (1981). *Statistical theory of reliability and life testing.* To Begin with, Silver Spring: MD.

Belzunce, F. (1999). On a characterization of right spread order by the increasing convex order. *Statistics and Probability Letters 45*, 103-110.

Cao, J. and Wang, Y. (1991). The NBUC and NWUC classes of life distributions. *Journal of Applied Probability 28*, 473-479.

Cox, D.R. and Oakes, D. (1984). *Analysis of Survival data.* New York: Chapman and Hall.

Deshpande, J. V. and Kochar, S. C. and Singh H. (1986). Aspects of positive ageing. *Journal of Applied Probability 23*, 748-758.

Fernandez-Ponce, J. M., Kochar, S. C. and Muñoz-Pérez, J. (1998). Partial orderings of distributions based upon right spread function. *Journal of Applied Probability 35*, 221-228.

Finkelstein, M. S. (2001). A note on some aging properties of the accelerated model. *Reliability Engineering and System Safty 71*, 109-112.

Kochar, S. C., Li, X. and Shaked, M. (2002). The total time on test transform and the excess wealth stochastic order of distributions. *Advances in Applied Probability 34*, 826-845.

Kochar, S. C. and Carrière, K. C. (1997) Connections among various variability orderings. *Statistics and Probability Letters 35*, 327–333.

Müller, A. and Stoyan D. (2002). *Comparison methods for stochastic models and risks.* John Wiley & Sons: New York.

Ross, S. M. (1996). *Stochastic Process.* 2nd edition, John Wiley & Sons: New York.

Shaked, M. and Shanthikumar, J. G. (1998). Two variability orders. *Probability in the Engineering and Informational Sciences 12*, 1-23.

Shaked, M. and Shanthikumar, J. G. (1994) *Stochastic Orders and Their Applications.* Academic Press: San Diego.